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Griffiths Singularity in the Random Ising Ferromagnet

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The explicit form of the Griffiths singularity in the random ferromagnetic Ising model in external magnetic field is derived. In terms of the continuous random temperature Ginzburg-Landau Hamiltonian it is shown that in the paramagnetic phase away from the critical point the free energy as the function of the external magnetic field h in the limit $h \to 0$ has the essential singularity of the form $\exp\left[-(const)/h^{D/3}\right]$ (where $1 < D < 4$ is the space dimensionality). It is demonstrated that in terms of the replica formalism this contribution to the free energy comes due to non-perturbative replica instanton excitations.

KEY WORDS: Quenched disorder; instantons; replicas; mean-field; non-analytic function; non-perturbative states.

1. INTRODUCTION

The history of the problem of the Griffiths singularities starts from the theorem of Lee and Yang [(1)] which states that the partition function of an (ordered) ferromagnetic Ising model (in any space dimensions and for any lattice connectivity) as the function of the external magnetic field h in the thermodynamic limit has a continuous distribution of zeros along the imaginary axis of the *complex* parameter $h = x + iy$. Moreover, in the paramagnetic phase depending on the temperature this distribution starts at finite distance Δy from the real axis, which means that here the free energy of the system is an analytic function of the *real* magnetic field h (with $y = 0$). On the other hand, when the temperature T approaches the phase transition point T_c from above the value of the interval Δy shrinks to zero, so that the distribution of zeros touches the real axes at $T = T_c$, which indicates (in agreement with the modern theory of the critical phenomena) that at the phase transition point the free energy must be a non-analytic function of (real) $h \to 0$.

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Now, if one would try to apply the above general observations for random systems, the consequences turn out to be much more tricky. As an example, let us consider bond diluted ferromagnetic Ising model, in which the critical temperature $T_c(p)$ is the (decreasing) function of the degree of dilution p. Here one can note that in an infinite system at a temperature T which is above $T_c(p)$ (such that the state of the system is paramagnetic) but below the critical temperature T_0 of the corresponding pure system, with a finite (exponentially small) probability there are exist arbitrary large less diluted ferromagnetic "islands" which are critical exactly at this given temperature T. It is important that such clusters exist at any temperature in the interval $T_c(p) < T < T_0$. Thus one can expect that the free energy of such random system must be a non-analytic function of the external magnetic field $h \to 0$ at *any* temperature between $T_c(p)$ and T_0 [(2)]. Unlike pure systems, however, here it is much more difficult to predict the explicit form of such non-analyticity.

For the one-dimensional diluted Ising chain it has been shown that its free energy $F(T, h)$ is non-analytic in the point $T = 0, h = 0$, and moreover the divergences of the pure system thermodynamics are replaced by an essential singularity at which all functions are finite and infinitely differentiable [(3)]. According to further more general studies (in terms of heuristic arguments [(4,5)] and the Bethe lattice systems [(4)]), the form of this singularity has been argued be of the type $\exp[-(const)/h]$.

At the same time there has been much interest in the dynamical properties of such systems in the temperature interval $T_c(p) < T < T_0$. It has been discovered that the relaxation processes here are slower than the exponential [(6)]. It turned out that due to the presence of rare large ferromagnetic clusters the relaxation of e.g. the order parameter takes either "stretched-exponential" form $\exp[-(t/\tau)^{\beta(T)}]$ controlled by the temperature dependent exponent $\beta < 1$ (the result preferred by the numerical simulations [(7)]), or even slower type of decay $\exp[-(const)(\ln t)^{D/(D-1)}]$ (predicted analytically [(6,8)]). Due to these quite non-trivial dynamical properties, which are essentially different from the paramagnetic phase, the state of the system in this temperature interval is usually called the Griffiths phase.

Besides dynamics, an essential progress has been achieved in the analytical investigation of the distribution of zeros of the partition function alone the imaginary axis of the complex magnetic field. In particular, formal replica supersymmetric calculations performed for the random temperature Ginzburg-Landau Hamiltonian [(9)] has demonstrated the importance of the instanton-like non-perturbative excitations which provide the development of the "tail" in the distribution of zeros in analogy with the density of states in the Anderson localization problem. On the other hand, the study of the diluted infinite-range Ising ferromagnet with *finite* connectivity in purely imaginary magnetic field $h = iy$ has shown that in the paramagnetic phase the tail of the density of zeros $\rho(y)$ has the explicit form of the type $\rho(y) \sim \exp[-f(T)/y]$, where the function of the temperature $f(T)$

vanishes in the critical point [10]. Similar tail has been also found numerically in the two-dimensional diluted Ising model [11]. All that, however, doesn't answer the main question, what is the explicit form of the Griffiths singularities of the thermodynamical quantities as the function of the *real* magnetic field in the point $h = 0$.

In this paper I am going to consider this problem in terms of the D-dimensional random temperature Ginzburg-Landau Hamiltonian (in dimensions $1 < D < 4$) in the paramagnetic phase away from the critical point. In the next section simple heuristic arguments will be proposed which demonstrate on a qualitative level the physical mechanism by which non-analytic contributions appear in the thermodynamical functions of such type of systems. After that, the systematic method of non-perturbative replica calculations will be formulated in Section III. Finally, in Section IV it will be demonstrated that non-analytic (Griffiths) contributions to the thermodynamics comes from non-perturbative instanton-like excitations. In the limit $h \to 0$ such contributions to the free energy is argued to have the explicit form

$$\Delta F \sim \exp\left[-C\, h^{-D/3}\right] \tag{1}$$

where the constant C is defined by the parameters of the Ginzburg-Landau Hamiltonian and by the strength of the disorder.

2. HEURISTIC ARGUMENTS

Before starting doing systematic calculations let us try to understand on a pure qualitative level, using simple "hand-waving-arguments," what is the physical mechanism by which non-analytic contributions are coming into the free energy (and others thermodynamical functions) in the paramagnetic phase of weakly disordered ferromagnetic Ising model in external magnetic field. Let us suppose that such system in D dimensions can be described by the continuous Ginzburg-Landau Hamiltonian

$$H = \int d^D x \left[\frac{1}{2}(\nabla\phi(x))^2 + \frac{1}{2}(\tau - \delta\tau(x))\,\phi^2(x) + \frac{1}{4}g\phi^4(x) - h\phi(x)\right] \tag{2}$$

Here, to ensure the existence of the ferromagnetic phase transition, the dimensionality D is assumed to be greater than one. Besides, the reduced temperature parameter τ is taken to be positive and sufficiently large to place the system into the paramagnetic phase. The disorder is modeled by a random function $\delta\tau(x)$ which is described by the Gaussian distribution,

$$P[\delta\tau] = p_0 \exp\left(-\frac{1}{4u}\int d^D x (\delta\tau(x))^2\right), \tag{3}$$

where u is the small parameter which describes the strength of the disorder, and p_0 is the normalization constant.

Intuitively, it is clear that non-trivial contributions to the thermodynamics are coming due to rare "ferromagnetic islands" in which the value of $\delta\tau(x)$ is on average bigger than τ. Let us consider such an island, which is characterized by the linear size L and the typical value of the "local temperature" $(\tau - \delta\tau) = -\xi < 0$ Its probability is exponentially small,

$$\mathcal{P}[L, \xi] \sim \exp\left(-\frac{(\tau + \xi)^2}{4u} L^D\right), \tag{4}$$

and therefore such islands are well separated from each other and can be considered as non-interacting. Note that the physical mechanism of slowing down of the relaxational processes due to the presence of these ferromagnetic islands are more or less clear. The magnetization orientation of the ferromagnetic cluster can be either "up" or "down," and if its size L is big, then it would require a long "elementary relaxation time" $t(L)$ to flip it from one orientation to the other (since flipping of the cluster would require overcoming a big energy barrier, which is proportional to L^{D-1}). The origin of the non-analytic contributions appearing in terms of pure statistical mechanics is much less clear: since time is formally infinite here, the presence of big energy barriers, separating the two orientations is irrelevant.

First, let us consider what is going on in the zero external magnetic field. Here one could distinguish two types of contributions to the thermodynamics:

(1) the perturbative one, coming from the spatial scales smaller than the correlation length R_c, which formally could be computed e.g. in terms of the renormalization-group (RG) approach [12];
(2) the non-perturbative contributions (missing in the RG treatment) due to the "up" and "down" ferromagnetic states of rare ferromagnetic islands discussed above, which are coming from the spatial scales bigger than the correlation length R_c.

It has to be noted that the island with small (negative) local temperature $-\xi$ can be characterized as having the distinct (mean-field) "up" and "down" states only if its size is much bigger than its local correlation length $R_c(\xi) \sim \xi^{-1/2}$ In what follows we are going to use the mean-field values of the critical exponents assuming that the temperature of the system is taken sufficiently far away from the critical region near T_c (the size of this region is of the order of $\tau_g = g^{2/(4-D)}$ which is small if the coupling parameter g is taken to be small).

According to eq. (2), the energy of a large ferromagnetic island is proportional to $-(\xi^2/g)L^D$. Since, according to eq. (4), the density of such islands is exponentially small, they can be considered as well separated and non-interacting with each other. Thus, their (averaged) contribution to the density of the free energy can be estimated as follows:

$$F_G \sim \int_0^\infty d\xi \int_{R_c(\xi)}^\infty dL \ \left(\frac{\xi^2}{g} L^D\right) \mathcal{P}[L, \xi]$$

$$\sim \frac{1}{g} \int_0^\infty d\xi \ \xi^2 \int_{R_c(\xi)}^\infty dL \ L^D \ \exp\left[-\frac{1}{4u}(\tau + \xi)^2 \ L^D\right]$$

$$\sim \frac{1}{g} \int_0^\infty d\xi \ \xi^{2-D/2} \ \exp\left[-(const)\frac{(\tau + \xi)^2}{u}\xi^{-D/2}\right] \tag{5}$$

Here in the integration over ξ (with the exponential accuracy) the leading contribution comes from the vicinity of the saddle-point value

$$\xi_* = \frac{D}{4 - D} \tau \tag{6}$$

(which is positive in dimensions $D < 4$, and $\xi_* \gg \tau_g$ provided $\tau \gg \tau_g$). In this way, with the exponential accuracy we obtain the following estimate for the non-perturbative contribution coming from rare ferromagnetic islands:

$$F_G \sim \exp\left[-(const)\frac{\tau^{(4-D)/2}}{u}\right] \tag{7}$$

In fact, this result (including the value of the $(const)$ factor), as we will see in section IV, can be derived analytically in terms of the formal replica calculations as the contribution from the localized (instanton-like) solutions of the mean-field saddle-point equations [13].

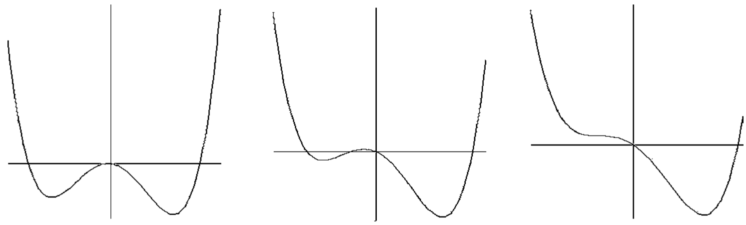
In the presence of non-zero external magnetic field h the situation becomes slightly more tricky. Let us consider again the ferromagnetic island of the size L with negative "local temperature" $(\tau - \delta\tau) = -\xi$ which is described by the Hamiltonian, eq. (2). Note first of all, that the effective potential

$$U(\phi) = -\frac{1}{2}\xi\phi^2 + \frac{1}{4}g\phi^4 - h\phi \tag{8}$$

has two minima only if the value of ξ is not too small, namely at

$$\xi > \xi_*(h) \sim h^{2/3} g^{1/3}, \tag{9}$$

otherwise, at $\xi < \xi_*(h)$, it has a unique minimum (see Fig.1). In the case of the two minima, the state "up" $(\phi > 0)$ has the lower energy $E_u(\xi, h, L)$, and it is the ground state of the island, while the state "down" $(\phi < 0)$ can be considered as the excitation since it has higher energy, $E_d(\xi, h, L)$. The crucial point is that the excited state, E_d disappears discontinuously at $\xi = \xi_*(h)$ such that the energy difference $\Delta E = E_d - E_u$ remains finite at this critical value of ξ.



(a) (b) (c)

Fig. 1. Qualitative shape of the potential $U(\phi)$, eq. (8)): (a) at $\xi \gg \xi_*(h)$; (b) at $\xi > \xi_*(h)$; (c) at $\xi = \xi_*(h)$.

Thus, the contribution to the total free energy of such ferromagnetic islands can be estimated as follows:

$$F_G(h) \sim -\int_{\xi_*(h)}^{\infty} d\xi \int_{R_c(\xi)}^{\infty} dL\ \mathcal{P}[L,\xi] \ln[e^{-E_u} + e^{-E_d}]$$
$$-\int_0^{\xi_*(h)} d\xi \int_{R_c(\xi)}^{\infty} dL\ \mathcal{P}[L,\xi] \ln[e^{-E_u}] \quad (10)$$

where $\mathcal{P}[L,\xi]$ is the probability to have the island of the size L with the local temperature ξ, eq. (4). Since the integration over L sticks to the lower bound R_c, we find:

$$F_G(h) \sim \int_0^{\infty} d\xi\ P(\xi)\ E_u(\xi,h)\ -\ \int_{\xi_*(h)}^{\infty} d\xi\ P(\xi) \ln[1 + e^{-\Delta E(\xi,h)}] \quad (11)$$

where

$$P(\xi)\ \sim\ \exp\left[-(const)\frac{(\tau+\xi)^2}{u}\xi^{-D/2}\right] \quad (12)$$

On the other hand, similar considerations for the zero field case (when $E_d = E_u$) yield

$$F_G(0) \sim -\int_0^{\infty} d\xi \int_{R_c(\xi)}^{\infty} dL\ \mathcal{P}[L,\xi] \ln\left[2e^{-E_u(\xi,0)}\right]$$
$$\sim \int_0^{\infty} d\xi\ P(\xi)\ [E_u(\xi,0) - (\ln 2)] \quad (13)$$

Thus, for the free energy difference, $\Delta F_G(h) = F_G(h) - F_G(0)$ we find:

$$\Delta F(h) \sim \int_0^{\infty} d\xi\ P(\xi)\ [E_u(\xi,h) - E_u(\xi,0)]$$
$$-\int_{\xi_*(h)}^{\infty} d\xi\ P(\xi) \ln\left[\frac{1+e^{-\Delta E(\xi,h)}}{2}\right] + (\ln 2)\int_0^{\xi_*(h)} d\xi\ P(\xi) \quad (14)$$

In the limit $h \to 0$ the first two terms in the above equation provide regular functions of h (both $(E_u(\xi, h) - E_u(\xi, 0))$ and $\Delta E(\xi, h)$ go to zero as a power functions of h in the limit $h \to 0$), while the last one is just the non-trivial Griffiths contribution $\delta F_G(h)$ which has the form of the essential singularity:

$$\delta F_G(h) \sim (\ln 2) \int_0^{\xi_*(h)} d\xi \; P(\xi) \sim \exp\left(-(const)\frac{\tau^2}{u}\xi_*^{-D/2}(h)\right) \tag{15}$$

Substituting here the value $\xi_*(h) = h^{2/3} g^{1/3}$, eq. (9), one eventually finds:

$$\delta F_G(h) \; \sim \exp\left(-(const)\frac{\tau^2}{u\, g^{D/6}}\; h^{-D/3}\right) \tag{16}$$

In Section IV it will be demonstrated how this result can be derived in terms of the formal replica calculations. But first we have to formulate the general lines of the replica approach for the non-perturbative contributions.

3. NON-PERTURBATIVE REPLICA CALCULATIONS

In this section I am going to formulate a general systematic approach for the calculations of non-perturbative contributions (if any) coming from local minima states, which in the configurational space are well separated from the ground state.

Let us consider a general random system described by a Hamiltonian $H[\phi(x)]$, and let us suppose that in addition to the ground state, there is another thermodynamically relevant (Griffith) region of the configurational space located "far away" from the ground state and separated from it by a finite barrier of the free energy (see Fig.2). In other words, we suppose that the partition function (of a given sample) can be represented in the form of two separate contributions:

$$Z = \int \mathcal{D}\phi(x) e^{-\beta H} \; = \; e^{-\beta F_0} + e^{-\beta F_1} \; \equiv \; Z_0 \; + \; Z_1 \tag{17}$$

where F_0 is the contribution coming from the vicinity of the ground state, and F_1 is the contribution of the Griffiths region. Then, for the averaged over disorder total free energy we find:

$$\mathcal{F} = -\frac{1}{\beta}\overline{\ln Z} = \overline{F_0} - \frac{1}{\beta}\overline{\ln\left[1 + Z_1 Z_0^{-1}\right]} \tag{18}$$

The second term here, which is just the Griffiths contribution, can be represented as follows:

$$F_G = -\frac{1}{\beta}\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}\overline{Z_1^m Z_0^{-m}} = -\frac{1}{\beta}\lim_{n\to 0}\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m} Z_n(m) \tag{19}$$

where

$$Z_n(m) = \prod_{b=1}^{m}\int \mathcal{D}\phi_b^{(1)} \prod_{c=1}^{n-m}\int \mathcal{D}\phi_c^{(0)}\; e^{-\beta H_n\left[\phi_1^{(1)},\ldots,\phi_m^{(1)},\phi_1^{(0)},\ldots,\phi_{n-m}^{(0)}\right]} \tag{20}$$

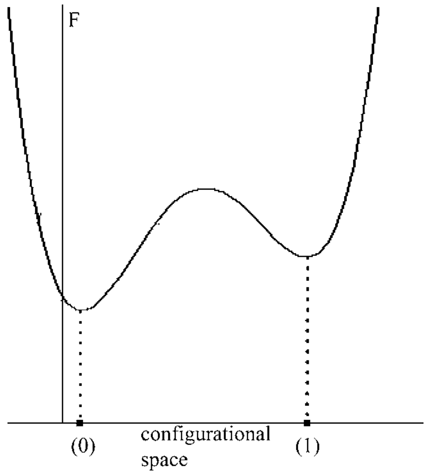


Fig. 2. Schematic structure of the free energy landscape in the case of two well separated thermodynamically relevant valleys of the configurational space

is the replica partition function ($H_n\,[\phi]$ is the corresponding replica Hamiltonian), in which the replica symmetry in the n-component vector field ϕ_a $(a = 1, ..., n)$ is assumed to be broken. Namely, it is supposed that the saddle-point equations

$$\frac{\delta H_n\,[\phi]}{\delta\phi_a(x)} = 0\,, \quad (a = 1, ..., n) \tag{21}$$

have non-trivial solutions with the RSB structure

$$\phi_a^*(x) = \begin{cases} \phi_1(x) \text{ for } a = 1, ..., m \\ \\ \phi_0(x) \text{ for } a = m+1, ..., n \end{cases} \tag{22}$$

with $\phi_1(x) \neq \phi_0(x)$, so that the integration in the above partition function, eq. (20), goes over fluctuations in the vicinity of these solutions:

$$\begin{aligned} \phi_b^{(1)}(x) &= \phi_1(x) + \varphi_b(x), \quad (b = 1, ..., m) \\ \phi_c^{(0)}(x) &= \phi_0(x) + \chi_c(x), \quad (c = 1, ..., n-m) \end{aligned} \tag{23}$$

It should be stressed that to be thermodynamically relevant, the RSB saddle-point solutions, eq. (22), should satisfy the following three crucial conditions:

(1) the solutions should be *local* in space, so that they are characterized by *finite* space sizes $R(m)$; in this case the partition function, eq. (20), will be proportional to the entropy factor $V/R^D(m)$ (where V is the volume of the system), and the corresponding free energy contribution F_G, eq. (19), will be extensive quantity;

(2) they should have *finite* energies $E(m) = H_n[\phi^*]$;
(3) the corresponding Hessian matrix of these solutions should have all eigenvalues positive.

Thus, in the systematic calculations one should find all saddle-point RSB solutions $\phi_a^*(x)$ (satisfying the above three requirements), eq. (22), after that one has to compute their energies $E(m)$ (for $n \to 0$), next one has to integrate over the fluctuations in the vicinity of these solutions, and finally one has to sum up the series

$$F_G = -\frac{V}{\beta} \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} R^{-D}(m)\, (\det \hat{T})_{n=0}^{-1/2}\, e^{-\beta E(m)} \tag{24}$$

where $\hat{T}$ is the $(n \times n)$ matrix

$$T_{aa'} = \frac{\delta^2 H[\phi]}{\delta\phi_a \delta\phi_{a'}}\Big|_{\phi=\phi^*} \tag{25}$$

The above scheme of calculations can be easily generalized for an arbitrary number of the Griffiths regions.

For example, let us consider the situation which is qualitatively represented in Fig.3, when in addition to the ground state, the system has *two* thermodynamically relevant Griffiths states (which is just the case for the considered random Ising

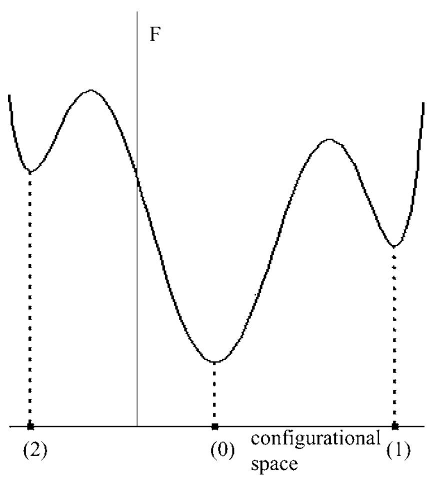


Fig. 3. Schematic structure of the free energy landscape in the case of three well separated thermodynamically relevant valleys of the configurational space

model). In this case instead of eq. (17) we will have

$$Z = \int \mathcal{D}\phi(x) e^{-\beta H} = e^{-\beta F_0} + e^{-\beta F_1} + e^{-\beta F_2} \equiv Z_0 + Z_1 + Z_2 \tag{26}$$

and correspondingly, instead of eq. (19) we find

$$\begin{aligned} F_G &= -\frac{1}{\beta} \overline{\ln\left[1 + Z_1 Z_0^{-1} + Z_2 Z_0^{-1}\right]} \\ &= -\frac{1}{\beta} \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \sum_{k=0}^{m} C_k^m \, \overline{\left(Z_1^k Z_2^{m-k} Z_0^{-m}\right)} \\ &= -\frac{1}{\beta} \lim_{n\to 0} \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \sum_{k=0}^{m} C_k^m \; Z_n(k,m) \end{aligned} \tag{27}$$

where $C_k^m = m!/(k!(m-k)!)$ is the combinatoric factor. Here, in the replica partition function

$$Z_n(k,m) = \prod_{b=1}^{k} \int \mathcal{D}\phi_b^{(1)} \prod_{c=1}^{m-k} \int \mathcal{D}\phi_c^{(2)} \prod_{d=1}^{n-m} \int \mathcal{D}\phi^{(0)} \, e^{-\beta H_n\left[\phi^{(1)},\phi^{(2)},\phi^{(0)}\right]} \tag{28}$$

the integration is supposed to be performed in the vicinity of the saddle-point replica vector

$$\phi_a^*(x) = \begin{cases} \phi_1(x), & \text{for } a = 1, ..., k \\ \phi_2(x), & \text{for } a = k+1, ..., m \\ \phi_0(x), & \text{for } a = m+1, ..., n \end{cases} \tag{29}$$

(where $\phi_1(x) \neq \phi_2(x) \neq \phi_0(x)$) which is the solution of the saddle-point equations (21). Finally, for the Griffiths contribution, instead of eq. (24) one obtain

$$F_G = -V \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{\beta m} \sum_{k=0}^{m} C_k^m \; R^{-D} (\det \hat{T})_{n=0}^{-1/2} e^{-\beta E(k,m)} \tag{30}$$

where $E(k,m) = H_{n\to 0}[\phi^*]$ is the energy of a given saddle-point solution, eq. (29), and $\hat{T}$ is the Hessian matrix, eq. (25).

It is worth noting that one can arrive to the same representations for the non-perturbative free energy contributions, eqs.(24) and (30), in terms of the so called vector replica symmetry breaking scheme [14,15], starting from the standard replica approach for random systems ($F = -\beta^{-1} \lim_{n\to 0}(\overline{Z^n} - 1)/n$)

In the next section we will implement the programme described above for the concrete case of weakly disordered ferromagnetic Ising model in the high temperature paramagnetic phase.

4. REPLICA INSTANTONS

After the standard Gaussian averaging over random $\delta\tau(x)$ (described by the distribution, eq. (3)) of the n-th power of the partition function one obtains the following replica Hamiltonian

$$H_n[\phi] = \int d^D x \left[\frac{1}{2} \sum_{a=1}^{n} (\nabla\phi_a)^2 + \frac{1}{2}\tau \sum_{a=1}^{n} \phi_a^2 + \frac{1}{4} g \sum_{a=1}^{n} \phi_a^4 - \frac{1}{4} u \sum_{a,b=1}^{n} \phi_a^2 \phi_b^2 - h \sum_{a=1}^{n} \phi_a \right] \tag{31}$$

The corresponding saddle-point equations are

$$-\Delta\phi_a(x) + \tau\phi_a(x) + g\phi_a^3(x) - u\phi_a(x) \left(\sum_{b=1}^{n} \phi_b^2(x) \right) = h \tag{32}$$

Substituting here the ansatz, eq. (29), the above saddle-point equations are reduced to

$$-\Delta\phi_i + \tau\phi_i + g\phi_i^3 - u\phi_i S = h \tag{33}$$

$(i = 1, 2, 0)$ where

$$S \equiv \sum_{a=1}^{n} \phi_a^2(x) \ = \ k\phi_1^2 + (m-k)\phi_2^2 + (n-m)\phi_0^2 \tag{34}$$

which in the limit $n \to 0$ turns into

$$S \ = \ k\phi_1^2 + (m-k)\phi_2^2 - m\phi_0^2 \tag{35}$$

Substituting the ansatz, eq. (29), into the Hamiltonian, eq. (31), for the energy of this configuration (in the limit $n \to 0$) we obtain:

$$E(k,m) = \int d^D x \left[\frac{k}{2} (\nabla\phi_1)^2 + \frac{(m-k)}{2} (\nabla\phi_2)^2 - \frac{m}{2} (\nabla\phi_0)^2 + U(\phi_1, \phi_2, \phi_0) \right] \tag{36}$$

where

$$U(\phi_1, \phi_2, \phi_0) = \frac{1}{2}\tau \left[k\phi_1^2 + (m-k)\phi_2^2 - m\phi_0^2\right] + \frac{1}{4} g \left[k\phi_1^4 + (m-k)\phi_2^4 - m\phi_0^4\right] - \frac{1}{4} u \left[k\phi_1^2 + (m-k)\phi_2^2 - m\phi_0^2\right]^2 - h\left[k\phi_1 + (m-k)\phi_2 - m\phi_0\right] \tag{37}$$

and the functions $\phi_1(x)$, $\phi_2(x)$ and $\phi_0(x)$ are defined by the equations (33) and (35).

4.1. Zero External Magnetic Field

First of all, we note in the case $h = 0$, due to the symmetry $\phi \to -\phi$ the solution of eqs.(33)–(35) takes the form:

$$\phi_1(x) = -\phi_2(x) \equiv \phi(x)$$
$$\phi_0(x) = 0 \tag{38}$$

where the function $\phi(x)$ is defined by the equation

$$-\Delta\phi(x) + \tau\phi(x) - \lambda(m)\phi(x)^3 = 0 \tag{39}$$

which is controlled by the parameter

$$\lambda(m) = um - g \tag{40}$$

In what follows this parameter will be assumed to be *positive*. In other words, the solution, which we are going to derived below, exists only for m such that

$$m \;>\; \left[\frac{g}{u}\right] \tag{41}$$

Substituting eqs.(38) into eq. (36)–(37) for the energy of this solution we obtain

$$E(k, m) \equiv E(m) = m \int d^D x \left[\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}\tau\phi^2 - \frac{1}{4}\lambda(m)\phi^4\right] \tag{42}$$

Note here, that one should not be confused by the "wrong" sign of the ϕ^4 term in the above expression (which for the usual field theory would indicate its absolute instability). In fact, as we will see below, the integration over the replica fluctuations around considered solution in the limit $n \to 0$ yields the Hessian matrix which has all the eigenvalues positive (this is quite standard situation for the replica theory: in the limit $n \to 0$, when the number of certain parameters become negative, everything turns "upside down," so that minima of the physical quantities turns into maxima of the corresponding replica quantities [15,16]).

Rescaling the fields,

$$\phi(x) = \sqrt{\frac{\tau}{\lambda(m)}}\psi(x/R_c(\tau)) \tag{43}$$

(where $R_c(\tau) = \tau^{-1/2}$), instead of eq. (39) one get the differential equation which contains no parameters:

$$-\Delta\psi(z) + \psi(z) - \psi^3(z) = 0 \tag{44}$$

Correspondingly, for the energy, eq. (42), one obtains:

$$E(m) = \frac{m}{um - g}\tau^{(4-D)/2}E_0 \tag{45}$$

where

$$E_0 = \int d^D z \left[\frac{1}{2}(\nabla\psi(z))^2 + \frac{1}{2}\psi^2(z) - \frac{1}{4}\psi^4(z) \right] \tag{46}$$

The equation (44) is well know in the field theory (see e.g. [17]): in dimensions $1 < D < 4$ it has the smooth (with $\psi'(0) = 0$) spherically symmetric instanton-like solution such that:

$$\begin{aligned} &\psi(z \leq 1) \sim \psi(0) \sim 1, \\ &\psi(z \gg 1) \sim e^{-z} \to 0. \end{aligned} \tag{47}$$

The energy E_0, eq. (46), of this solutions is a finite and *positive* number. In particular, at $D = 3$, $\psi_0 \simeq 4.34$ and $E_0 \simeq 18.90$. Note that according to the rescaling, eq. (43), the size of the instanton solution in terms of the original fields $\phi(x)$ is $R_c = \tau^{-1/2}$. This size does not depends on k and m, and it coincides with the usual mean-field correlation length of the Ginsburg-Landau theory. Note also that due to the symmetry $\phi \to -\phi$ of the considered solution its parameters do not depend on k. Thus, we can perform the summation over k in the series, eq. (30), which yields the trivial factor 2^m:

$$\begin{aligned} F_G \simeq -V R_c(\tau)^{-D} \sum_{m>[g/u]}^{\infty} \frac{(-1)^{m-1}}{m} 2^m (\det \hat{T})_{n=0}^{-1/2} \\ \times \exp\left(-E_0 \frac{m}{um - g} \tau^{(4-D)/2}\right) \end{aligned} \tag{48}$$

In other words, the considered two-step structure, eq. (29)–(30), is equivalent to the one-step ansatz, eq. (22),

$$\phi_a^*(x) = \begin{cases} \sqrt{\frac{\tau}{\lambda(m)}}\psi(x\sqrt{\tau}) & \text{for } a = 1, ..., m \\ 0 & \text{for } a = m+1, ..., n \end{cases} \tag{49}$$

which has additional degeneracy factor 2^m.

The final step is the integration over fluctuations which define the Hessian factor ($\det \hat{T}$). Introducing small fluctuations $\varphi_a(x)$ near the above instanton solution, $\phi_a(x) = \phi_a^*(x) + \varphi_a(x)$, in the Gaussian approximation we get the following Hamiltonian for the fluctuating fields:

$$H[\varphi] \simeq \int d^D x \left[\frac{1}{2}\sum_{a=1}^{n}(\nabla\varphi_a)^2 + \frac{1}{2}\tau \sum_{a,a'=1}^{n} K_{aa'}(x)\varphi_a\varphi_{a'} \right] \tag{50}$$

where the matrix $K_{aa'}(x)$ contains the $m \times m$ block:

$$K_{bb'}^{(m)}(x) = \left(1 - \frac{um - 3g}{um - g}\psi^2(x\sqrt{\tau})\right)\delta_{bb'} - \frac{2u}{um - g}\psi^2(x\sqrt{\tau}) \tag{51}$$

$(b, b' = 1, ..., m)$ and the diagonal elements for the remaining $(n - m)$ replicas:

$$K_{cc'}^{(n-m)} = \left(1 - \frac{um}{um - g}\psi^2(x\sqrt{\tau})\right)\delta_{cc'} \tag{52}$$

$(c, c' = m + 1, ..., n)$. To obtain the explicit result for the Hessian, let us approximate the instanton solution, eq. (47), by the θ-like function

$$\psi(z) = \begin{cases} \psi_0, & \text{for } 0 \le z \le 1 \\ 0, & \text{for } z > 1 \end{cases} \tag{53}$$

Then, the approximate Hamiltonian for the fluctuating fields takes much more simple form

$$H[\varphi] \simeq \frac{1}{2}\sum_{a,a'=1}^{n}\int_{|p|\gg\sqrt{\tau}} \frac{d^D p}{(2\pi)^D}\,[p^2\delta_{aa'} + \tau K_{aa'}]\varphi_a(p)\varphi_{a'}(-p)$$
$$+\frac{1}{2}\sum_{a=1}^{n}\int_{|p|\ll\sqrt{\tau}} \frac{d^D p}{(2\pi)^D}\,p^2|\varphi_a(p)|^2 \tag{54}$$

Here the p-independent matrix $K_{aa'}$ is given by eqs. (51)–(52), where instead of the function $\psi(x\sqrt{\tau})$ one has to substitute the constant ψ_o.

The integration over the modes with momenta $p \ll \sqrt{\tau}$ (corresponding to the scales much bigger than the size of the instanton), which are described by the second term of the above Hamiltonian, gives the contribution of the form $\exp(-nVf_{RS})$. This contribution is irrelevant in the limit $n \to 0$, which is quite natural, because all such contributions must be already contained in the perturbative part of the free energy F_0, eq. (18), which we do not consider here.

The integration over the modes with momenta $p \gg \sqrt{\tau}$ is slightly cumbersome but straightforward:

$$(\det \hat{T})^{-1/2} \simeq \exp\left[-\frac{\tau^{-D/2}}{2}\int_{p\gg\sqrt{\tau}} \frac{d^D p}{(2\pi)^D}\ \mathrm{Tr}\ \ln\left(p^2\delta_{aa'} + \tau K_{aa'}\right)\right] \tag{55}$$

The matrix under the logarithm in the above equation contains $(m - 1)$ eigenvalues:

$$\lambda_1 = p^2 + \tau\left(1 - \frac{um - 3g}{um - g}\psi_o^2\right) \tag{56}$$

one eigenvalue:

$$\lambda_2 = p^2 + \tau\left(1 - \frac{um - 3g}{um - g}\psi_o^2\right) - \tau\frac{2um}{um - g}\psi_o^2 \tag{57}$$

and $(n - m)$ eigenvalues:

$$\lambda_3 = p^2 + \tau\left(1 - \frac{um}{um - g}\psi_o^2\right) \tag{58}$$

which are all positive. Substituting these eigenvalues into eq. (55), after simple algebra in the limit $n \to 0$ one eventually obtains the following result:

$$(\det \hat{T})^{-1/2} \simeq \exp\left[\frac{3m}{2(um-g)} g\psi_o^2\right] \tag{59}$$

Finally, substituting this value into eq. (48) we find

$$F_G \simeq -V R_c(\tau)^{-D} \sum_{m>[g/u]}^{\infty} \frac{(-1)^{m-1}}{m} 2^m \times \exp\left[-E_0 \frac{m}{um-g} \tau^{(4-D)/2} + \frac{3m}{2(um-g)} g\psi_o^2\right] \tag{60}$$

Here one can note that under condition

$$\tau \gg \tau_g = g^{2/(4-D)} \tag{61}$$

the second term in the exponential of eq. (60) (which is the fluctuations contribution) can be neglected compared to the first one. This is not surprising because eq. (61) is nothing else, but the familiar Ginzburg-Landau condition which defines the temperature region away from T_c, where the critical fluctuations are irrelevant, and the behavior of the system is described by the mean-field exponents.

The exact summation of the series in eq. (60) is rather tricky problem, but with the exponential accuracy it can be estimated in a very simple way. One can easily see that in the limit of weak disorder, at $u \ll g$, the leading contribution in this summation comes from the region $m \gg g/u \gg 1$ (where the exponential factor in eq. (60) becomes m-independent) and this contribution is

$$F_G \sim \exp\left(-E_0 \frac{\tau^{(4-D)/2}}{u}\right) \tag{62}$$

We see that this result nicely coincides with the naive "hand-waving" estimate, eq. (7), where the (*const*) factor is the instanton energy E_0 (in three dimensions $E_0 \simeq 18.9$).

4.2. Non-zero External Magnetic Field

Technically, the situation in non-zero magnetic field becomes much more cumbersome, but on a qualitative level, the main idea of the approach remains very simple. According to the physical discussion of Section II, non-analytic contribution to the free energy in the presence of external magnetic field appears due to the fact, that some of the instanton-like configurations (of the type, considered in the previous subsection) disappear via a finite jump.

To understand that, let us consider first the structure of the potential energy $U(\phi_1, \phi_2, \phi_0)$, eq. (37). The extrema of this potential are defined by the three equations

$$\tau\phi_i + g\phi_i^3 - u\phi_i S = h \tag{63}$$

($i = 1, 2, 0$), where

$$S \;=\; k\phi_1^2 + (m-k)\phi_2^2 - m\phi_0^2 \tag{64}$$

Simple algebraic manipulations reduce these equations to

$$\phi_1 + \phi_2 + \phi_0 = 0 \tag{65}$$

$$\phi_1\phi_2(\phi_1 + \phi_2) = -\frac{h}{g} \tag{66}$$

and

$$\tau + [g + u(m-k)]\phi_1^2 + (g + uk)\,\phi_2^2 + (g + 2um)\,\phi_1\phi_2 = 0 \tag{67}$$

First, treating the magnetic field h here as a small correction to the zero-field solution,

$$\phi_1^{(h=0)} = -\phi_2^{(h=0)} \equiv \phi(m) = \sqrt{\frac{\tau}{um - g}}$$

$$\phi_0^{(h=0)} = 0 \tag{68}$$

one easily finds

$$\phi_1 \simeq \phi(m) + \frac{h}{g\phi^2(m)} + O\left(h^2\right),$$

$$\phi_2 \simeq -\phi(m) + O\left(h^2\right),$$

$$\phi_0 \simeq -\frac{h}{g\phi^2(m)} + O\left(h^2\right) \tag{69}$$

It is clear that this linear in h shift of the extremum of the potential $U(\phi_1, \phi_2, \phi_3)$ provide not more than a linear in h corrections to the zero field instanton space configuration as well as to its free energy contribution considered in the previous subsection.

It has to be noted, however, that the above result, eq. (69), is valid only until the summation parameters k and m are not too large:

$$m, k \;\ll\; \frac{\tau}{u}\left(\frac{g}{h}\right)^{2/3} \tag{70}$$

Simple analysis shows that as k and m grow, the values of (negative) $\phi_2(k, m)$ and $\phi_0(k, m)$ become closer and closer to each other, and finally, one arrive to the critical configuration when their values coincide. Substituting $\phi_0 = \phi_2 \equiv \phi_2^{(cr)}$

into equations (65)–(67) one easily finds that

$$\phi_2^{(cr)} = -\left(\frac{h}{2g}\right)^{1/3}$$
$$\phi_1^{(cr)} = 2\left(\frac{h}{2g}\right)^{1/3} \tag{71}$$

and this critical configuration (in the limit $h \to 0$) takes place at

$$k_c(h) \simeq \frac{\tau}{3u}\left(\frac{2g}{h}\right)^{2/3} \tag{72}$$

for arbitrary value of $m \geq k_c$.

At larger values of k the system of equations (65)–(67) have no solutions at all. The simplest way to understand this, is to consider the eqs.(66) and (67) for k and m much bigger than k_c. In this case eq. (67) would require the values of ϕ_1 and ϕ_2 as the functions of k and m to be of order $k^{-1/2}$ and $m^{-1/2}$, tending to zero as $m, k \to \infty$, while eq. (66) tells that both ϕ_1 and ϕ_2 must remain finite.

Thus, the summations in the instanton free energy contribution, eq. (30), has to be limited by the finite value $k_c(h)$:

$$F_G(h) = -V\sum_{m=m_0}^{k_c}\frac{(-1)^{m-1}}{m}\sum_{k=0}^{m} C_k^m \frac{e^{-E(k,m;h)}}{R^D(\det \hat{T})_{n=0}^{1/2}}$$
$$-V\sum_{m=k_c}^{\infty}\frac{(-1)^{m-1}}{m}\sum_{k=0}^{k_c} C_k^m \frac{e^{-E(k,m;h)}}{R^D(\det \hat{T})_{n=0}^{1/2}} \tag{73}$$

(here $m_0 = [g/u] + 1$). This expression has to be compared with the corresponding summation at $h = 0$, studied in the previous subsection. Although all the terms in these series are non-analytic functions of the disorder parameter u (see previous subsection), until $k, m \ll k_c(h)$, the difference between the terms with $h \neq 0$ and the corresponding terms with $h = 0$ can be represented in the form of corrections in powers of h. On the other hand, as we will see below, the terms with $k \sim k_{cr}(h)$ are non-analytic functions of h, and their differences with the corresponding zero-field contributions (in particular the differences of the instanton energies) can not be expended in powers of h. It is these terms, which are the contributions of the critical instantons which yield the non-analytic in h part of the free energy $\delta F_G(h)$. Rigorous extracting of this piece must be a very difficult problem (it would require derivation of the instantons energy for arbitrary k, m and h, as well as summations of the full series in eq. (73)), but with the exponential accuracy the form of this non-analytic singularity is defined only by the energy $E^{(cr)}(h)$ of the critical instanton,

$$\delta F_G(h) \sim \exp\left[-E^{(cr)}(h)\right] \tag{74}$$

Let us study the instanton configuration and estimate its energy in the vicinity of its critical state. For that let us rescale the fields,

$$\phi_i = \left(\frac{h}{g}\right)^{1/3} \psi_i\,(x/R_c) \tag{75}$$

$(i = 1, 2, 0)$ where

$$R_c = h^{-1/3} g^{-1/6} \tag{76}$$

defines the spacial size of the critical instanton. In terms of the rescaled fields ψ, the energy of the instanton takes the form

$$E(k, m) = \frac{R_c^D h^{4/3}}{g^{1/3}} \int d^D z \left[\frac{k}{2}\,(\nabla\psi_1)^2 + \frac{(m-k)}{2}\,(\nabla\psi_2)^2 - \frac{m}{2}\,(\nabla\psi_0)^2 + +U\,(\psi_1, \psi_2, \psi_0)\right] \tag{77}$$

where

$$\begin{aligned} U\,(\psi_1, \psi_2, \psi_0) = &\ \frac{1}{2}\tilde{\tau}\,[k\psi_1^2 + (m-k)\psi_2^2 - m\psi_0^2] \\ &+ \frac{1}{4}\,[k\psi_1^4 + (m-k)\psi_2^4 - m\psi_0^4] \\ &- \frac{1}{4}\tilde{u}\,[k\psi_1^2 + (m-k)\psi_2^2 - m\psi_0^2]^2 \\ &- \ [k\psi_1 + (m-k)\psi_2 - m\psi_0] \end{aligned} \tag{78}$$

is the potential controlled by two rescaled parameters

$$\begin{aligned} \tilde{\tau} &= \tau g^{-1/3} h^{-2/3} \\ \tilde{u} &= \frac{u}{g} \end{aligned} \tag{79}$$

The instanton configuration is defined by the three equations,

$$-\Delta\psi_i + \tilde{\tau}\psi_i + \psi_i^3 - \tilde{u}\psi_i S = 1 \tag{80}$$

$(i = 1, 2, 0)$ where

$$S \ = \ k\psi_1^2 + (m-k)\psi_2^2 - m\psi_0^2 \tag{81}$$

Let us consider these equations in the limit $h \to 0$ and and for the values of k and m of order of $k_c \sim h^{-2/3} \to \infty$. There are two types of terms in eqs.(80): (1)

$\tilde{\tau}\psi_i$ and $\tilde{u}\psi_i S$ which are both of order $h^{-2/3} \to \infty$; and (2), the rest of the terms, which are of order of one. In the limit $h \to 0$ the diverging terms must balance each other:

$$\tilde{\tau}\psi_i \sim \tilde{u}\left(k\psi_1^2 + (m-k)\psi_2^2 - m\psi_3^2\right)\psi_i \tag{82}$$

This condition is consistent with the requirement that both k and m are of the order of $k_c(h)$, eq. (72).

Substituting these estimates into eqs.(77), (78), for the energy of the instanton in its critical configuration we get

$$E^{(cr)} \sim \frac{\tau^2}{ug^{D/6}} h^{-D/3} \tag{83}$$

Thus, with the exponential accuracy, the field dependent non-analytic part of the free energy has the following explicit form

$$\delta F_G(h) \sim e^{-E^{(cr)}} \sim \exp\left(-(const)\frac{\tau^2}{ug^{D/6}} h^{-D/3}\right) \tag{84}$$

which perfectly agree with the mean-field "hand-waving" estimate, eq. (16). It has to be noted, however, that in deriving the above result we have neglected the effects of the critical fluctuations, which is justified only if we consider the system at temperatures not too close to the critical point, $\tau \gg \tau_g = \tau^{2/(4-D)}$. In the close vicinity of T_c, at $\tau \ll \tau_g$, the situation becomes much more complicated: here one would have to combine the systematic (renormalization-group) integration over fluctuation with the background instanton solutions.

5. CONCLUSIONS

In this paper the systematic approach for the non-perturbative calculations in disordered systems has been formulated (Section III). It has been demonstrated how the proposed scheme works in the most simple but non-trivial case of weakly disordered ferromagnetic Ising model away from its critical region. Here the non-analytic (as the functions of the disorder parameter and the external magnetic field) contributions to the free energy has been derived, eqs.(62), (84), and it has been demonstrated that in terms of the replica field theory such contributions appear due to instanton-like excitations. Of course, it is hardly possible to register the presence of these exponentially small parts of the free energy both in real and in numerical experiments, and in this sense the present results has mostly pure theoretical interest. On the other hand, thinking about the others much more complicated problems of the statistical mechanics of disordered systems, the investigations made in this paper look rather promising. In particular, it does not look completely unrealistic to try to combine the present non-perturbative scheme with the renormalization-group treatment of the critical fluctuations, to settle down recent suspicion [18] that non-perturbative degrees of freedom could be quite relevant in the vicinity of the critical point, so that the nature of the phase transition in random ferromagnetic systems may appear to be not as simple as it was thought in early days of the theory of the critical phenomena in disordered materials. [12]

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